Integration Bee Formula booklet

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Standard integrals (+C omitted)

Trigonometric and Hyperbolic ($a > 0$):

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) \qquad \left[= \frac{\pi}{2} - \arccos\left(\frac{x}{a}\right)\right]$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{x^2 + a^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \qquad \left[= \frac{\pi}{2a} - \frac{1}{a}\operatorname{arccot}\left(\frac{x}{a}\right)\right]$$

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)$$

Integral (and summation) functions:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t \qquad \text{(equivalent to } (z-1)! \text{ if } z \in \mathbb{N}^+\text{)}$$

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)} = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,\mathrm{d}t \qquad (z_1, z_2 > 0)$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (s > 1)$$

Constants and specific values

Specific values:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\zeta(2) = \frac{\pi^2}{6}$$

Constants (note $\lfloor x \rfloor$ is x rounded down to the nearest integer):

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \frac{\pi}{2} \qquad \text{(Wallis' Product)}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G \qquad \text{(Catalan's Constant)}$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right) = \gamma \qquad \text{(Euler-Mascheroni's Constant)}$$

Formulae and identities

Expansions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1)!}$$

$$\operatorname{arsinh}(x) = \sum_{n=0}^{\infty} \left(\frac{(2n)!(-1)^n}{2^{2n}(n!)^2} \right) \frac{x^{2n+1}}{2n+1}$$

$$\arcsin(x) = \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^n(n!)^2} \right) \frac{x^{2n+1}}{2n+1}$$

$$\operatorname{arcosh}(x) = \ln(2x) - \sum_{n=1}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \right) \frac{x^{-2n}}{2n}$$

$$\arccos(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^n(n!)^2} \right) \frac{x^{2n+1}}{2n+1}$$

$$\operatorname{artanh}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{2n+1}$$

Inverse hyperbolic identities:

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\operatorname{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x} \right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right)$$

Facts about Γ, for $(z \in \mathbb{C})$:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\Gamma(z+1) = z\Gamma(z)$$

Finite sums:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

Weierstrass Substiution, $t = \tan\left(\frac{x}{2}\right)$:

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin(x) = \frac{2t}{1+t^2}$$

$$\tan x = \frac{2t}{1-t^2}$$